Generalized synchronization of chaos: The auxiliary system approach

Henry D. I. Abarbanel,[1,2] Nikolai F. Rulkov,[2] and Mikhail M. Sushchik[3]
[1]Department of Physics and Marine Physical Laboratory, Scripps Institution of Oceanography, University of California, San Diego, La Jolla, California 92093-0402
[2]Institute for Nonlinear Science, University of California, San Diego, La Jolla, California 92093-0402
[3]Department of Physics and Institute for Nonlinear Science, University of California, San Diego, La Jolla, California 92093-0402

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Synchronization of chaotic oscillators in a generalized sense leads to richer behavior than identical chaotic oscillations in coupled systems. It may imply a more complicated connection between the synchronized trajectories in the state spaces of coupled systems. We suggest a method here that can be used to detect and study generalized synchronization in drive-response systems. This technique, *the auxiliary system method*, utilizes a second, identical response system to monitor the synchronized motions. The method can be implemented both numerically and experimentally and in some cases it leads to analytical results for generalized synchronization. [S1063-651X(96)02505-6]

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I. INTRODUCTION

Synchronization of chaos is a striking behavior of coupled nonlinear systems with chaotic uncoupled behavior. This behavior appears in many physical and biological processes and it may be responsible for the transition to low-dimensional behavior in systems with many degrees of freedom [1, 2]. It would seem to play an important role in the ability of complex nonlinear oscillators, such as neurons, to cooperatively act in the performance of various functions [3].

Synchronization of chaos is often understood as a regime in which two coupled chaotic systems exhibit identical, but still chaotic, oscillations [4–6]. In the case of synchronization of drive-response systems this regime of identical oscillations only occurs at a certain point in the parameter space of a response system and thus represents a rather degenerate case. This fact may pose a problem for using the results of theoretical analyses in practical applications of synchronized chaos. It was shown in [5, 7–9] that when the parameters of the coupled systems are detuned from the point where oscillations are identical, the coupled systems can still remain synchronized in a generalized sense; namely, the projections of synchronized trajectories onto partial state spaces of the coupled systems are connected by a continuous transformation. In our earlier paper [7] we introduced a class of chaotic synchronized motions in drive-response systems, which we called ''generalized synchronization of chaos.'' Since the transformation between drive and response dynamical variables that embodies the generalized synchronization can be very complicated, one needs special methods to detect the existence of the transformation and study this kind of synchronous behavior. Some numerical tools for the detection of generalized synchronization in systems with unidirectional coupling were developed and used in [7, 9].

In this paper we present another method that in some cases can be used for detection and characterization of forced generalized synchronization. This technique, which we call *the auxiliary system approach*, is particularly appealing since it can be implemented directly in an experiment without using any computational power. This is in contrast to the tools described in [7, 9]. In addition, as we shall show below, the auxiliary systems method allows one to utilize analytical approaches for studying generalized synchronization.

In Sec. II we give a general description of the auxiliary system method for detecting generalized synchronization of chaos. In Sec. III we demonstrate the implementation of this approach in our theoretical analysis of synchronized chaos using the example of synchronization of the chaos in a Lorenz system with a chaotic drive signal taken from a Rössler system. In Sec. IV we present the results of an experiment with synchronization of chaos in nonlinear electrical circuits with different parameters. The auxiliary system method is used for detection of chaotic synchronization between the circuits. We exhibit experimental results for both the synchronized and unsynchronized motions of the circuits.

II. THE AUXILIARY SYSTEM APPROACH FOR GENERALIZED SYNCHRONIZATION

We work with nonlinear systems composed of an *autonomous drive* system with the dynamical variables $\mathbf{x}$ in a phase space X coupled into a *response* system with dynamical variables $\mathbf{y}$ in the state space Y. The dynamics of the drive $\mathbf{x}(t)$ and response $\mathbf{y}(t)$ system are

$$\frac{d\mathbf{x}(t)}{dt}=\mathbf{F}(\mathbf{x}(t)), \tag{1}$$

$$\frac{d\mathbf{y}(t)}{dt}= \mathbf{G}(\mathbf{y}(t),\mathbf{g},\mathbf{x}(t)). \tag{2}$$

The coupling of the response system to the drive is characterized by the set of parameters $\mathbf{g}$. We assume that when $\mathbf{g}=0$, $\mathbf{G}(\mathbf{y},0,\mathbf{x})$ is independent of the drive variables $\mathbf{x}$ and both the drive and response systems evolve on separate chaotic attractors. The chaotic dynamics of the drive system does not depend on the parameters of the response system, so the connection between the systems is unidirectional.

In the spirit of our earlier paper [7] we use the following definition of generalized synchronization for the systems (1) and (2). When $\mathbf{g}\neq\mathbf{0}$, we say that the chaotic oscillations in the two systems are synchronized in a generalized sense if there is a transformation $\boldsymbol{\phi}$: $X\rightarrow Y$ that takes the trajectories of the attractor in X space into the trajectories of the attractor in the Y space, so that $\mathbf{y}(t)=\boldsymbol{\phi}(\mathbf{x}(t))$, and if this transformation does not depend upon the initial conditions of the response system $\mathbf{y}(0)$ in the basin of attraction of the synchronized attractor. We emphasize that in this definition of generalized synchronization *the existence of transformation* $\boldsymbol{\phi}$ *is required only for the trajectories on the attractor*. The transformation is not required to exist for the transient trajectories.

In this paper we consider a class of generalized synchronized motions for which the transformation $\boldsymbol{\phi}(\cdot)$ has the following properties.

Property 1. $\boldsymbol{\phi}(\cdot)$ has no explicit time dependence.

Property 2. On the synchronized attractor it transforms points in the X space into *points* (not continuous domains) in the Y space. The transformation is not required to preserve the number of points operated upon. Thus the transformation is allowed to has finite number of branches with a defined rule of transition from one branch to another [10].

Property 3. On each branch the transformation is locally continuous [11].

The transformation associated with *synchronized* motions on the overall chaotic attractor in the total $X\oplus Y$ phase space is $\mathbf{y}(t)=\boldsymbol{\phi}(\mathbf{x}(t))$. The existence of a transformation $\boldsymbol{\phi}(\cdot)$ guarantees the ability to predict the state of the response system from measurements of $\mathbf{x}(t)$ alone, *once transients die out*. Again we emphasize that this relation between the drive and response dynamical variables need not hold everywhere in the system phase space, but need hold only on the attractor. The predictability of the chaotic behavior of the response system from time series generated by the drive system was used in [7] for detecting of the presence of generalized synchronization of chaos. In this paper we propose an alternative method to test the predictability of the response system from knowledge of $\mathbf{x}(t)$ and, in so doing, to detect the synchronous chaotic behavior in nontrivial cases.

We consider the *auxiliary system*

$$\frac{d\mathbf{z}(t)}{dt}=\mathbf{G}(\mathbf{z}(t),\mathbf{g},\mathbf{x}(t)), \tag{3}$$

which is *identical* to the response system. Clearly, when the response (2) and auxiliary (3) systems are driven by the same signal $\mathbf{x}(t)$, then the vector fields in the phase spaces of the response and auxiliary systems are identical and the systems can evolve on identical attractors. Indeed, one may ask how the orbits $\mathbf{y}(t)$ and $\mathbf{z}(t)$ could fail to be on identical attractors. The answer lies in the possibility that there are several basins of attraction for the driven system with vector field $\mathbf{G}(\cdot,\mathbf{g},\mathbf{x}(t))$, for if $\mathbf{y}(0)$ and $\mathbf{z}(0)$ lie in different basins of attraction, one will see quite different orbits. However, if these initial conditions lie in the same basin of attraction, after some transients die out, we will certainly have $\mathbf{z}(t)$ and $\mathbf{y}(t)$ moving around the same geometrical object in phase space.

Under special circumstances, it could be that $\mathbf{z}(t)$ and $\mathbf{y}(t)$ will not only lie on the same attractor, but they could themselves be identical, namely, $\mathbf{y}(t)=\mathbf{z}(t)$. In general, this will not be the case, for two orbits on the same chaotic attractor move apart exponentially rapidly until they are of order the size of the attractor apart and then they remain uncorrelated as they continue to develop. However, if the systems are each synchronized to the drive variables $\mathbf{x}(t)$ through the generalized synchronization relation $\mathbf{y}(t)=\boldsymbol{\phi}(\mathbf{x}(t))$ and $\mathbf{z}(t)=\boldsymbol{\phi}(\mathbf{x}(t))$, then it is clear that a solution exists in the form $\mathbf{y}(t)=\mathbf{z}(t)$. The stability of the synchronization manifold where $\mathbf{y}(t)=\boldsymbol{\phi}(\mathbf{x}(t))$ ensures us that $\mathbf{z}(t)$ is able to track $\mathbf{y}(t)$ as $\mathbf{z}(t)=\mathbf{y}(t)$ in a stable manner too.

In general, then, the auxiliary system is just another response system and in the absence of generalized synchronized motion of the response to the drive, the orbits of the response system and the auxiliary system will share the same complicated attractor but will be otherwise unrelated. In the case of generalized synchronization, there is a stable regime of oscillations where the orbits of the response system and the orbits of the auxiliary system become identical after transients die out and we observe the simple identity relationship $\mathbf{y}(t)=\mathbf{z}(t)$. The stable regime of these identical oscillations guarantees the possibility of prediction of the current state of the response system, given the history of evolution of the drive system, and therefore indicates the presence of the generalized synchronization. The identity $\mathbf{y}(t)=\mathbf{z}(t)$ is a much simpler relationship to test for than the unknown, generically complicated generalized synchronization relationship $\mathbf{y}(t)=\boldsymbol{\phi}(\mathbf{x}(t))$.

It is easy to show that the linear stability of the manifold $\mathbf{z}(t)=\mathbf{y}(t)$ is equivalent to the linear stability of the manifold of synchronized motions in $X\oplus Y$, which is determined by $\boldsymbol{\phi}(\cdot)$. The linearized equations that govern the evolution of the quantities $\boldsymbol{\xi}_y(t)=\mathbf{y}(t)-\boldsymbol{\phi}(\mathbf{x}(t))$ and $\boldsymbol{\xi}_z(t)=\mathbf{z}(t)-\boldsymbol{\phi}(\mathbf{x}(t))$ are

$$\frac{d\boldsymbol{\xi}_y(t)}{dt}=\mathbf{DG}(\boldsymbol{\phi}(\mathbf{x}(t)),\mathbf{g},\mathbf{x}(t))\cdot\boldsymbol{\xi}_y(t), \tag{4}$$

$$\frac{d\boldsymbol{\xi}_z(t)}{dt}=\mathbf{DG}(\boldsymbol{\phi}(\mathbf{x}(t)),\mathbf{g},\mathbf{x}(t))\cdot\boldsymbol{\xi}_z(t), \tag{5}$$

where

$$\mathbf{DG}(\mathbf{w},\mathbf{g},\mathbf{x}(t))=\frac{\partial\mathbf{G}(\mathbf{w},\mathbf{g},\mathbf{x}(t))}{\partial\mathbf{w}}. \tag{6}$$

Since the linearized equations for $\boldsymbol{\xi}_y(t)$ and $\boldsymbol{\xi}_z(t)$ are identical, the linearized equations for $\boldsymbol{\xi}_z(t)-\boldsymbol{\xi}_y(t)=\mathbf{z}(t)-\mathbf{y}(t)$ have the same Jacobian matrix $\mathbf{DG}(\cdot,\mathbf{g},\mathbf{x}(t))$ as in the previous equation. Therefore, if the manifold of synchronized motions in $X\oplus Y\oplus Z$ is linearly stable for $\mathbf{z}(t)-\mathbf{y}(t)$, then it is linearly stable for $\boldsymbol{\xi}_y(t)=\mathbf{y}(t)-\boldsymbol{\phi}(\mathbf{x}(t))$ and vice versa. Note that the linearized equation for $\mathbf{z}(t)-\mathbf{y}(t)$ is identical to the equation that defines the conditional

Lyapunov exponents [6] for the response system. Thus, when the manifold $\mathbf{z}=\mathbf{y}$ is linearly stable, the conditional Lyapunov exponents for the response system, conditioned on the value of the drive $\mathbf{x}(t)$, are all negative.

We have thus demonstrated that to study the transition to generalized synchronized chaos, the analysis of stability of the synchronization manifold in the space $X \oplus Y$, which in general may have a very complicated shape $\mathbf{y}(t)=\boldsymbol{\phi}(\mathbf{x}(t))$, can be replaced by the analysis of the stability of the quite simple manifold $\mathbf{z}(t)=\mathbf{y}(t)$ in $Z \oplus Y$ space.

The observation of a locally stable regime of identical oscillations in the response and auxiliary systems guarantees the existence of the transformation $\boldsymbol{\phi}(\cdot)$ satisfying properties (1)–(3) noted above. Indeed, suppose the transformation $\boldsymbol{\phi}(\cdot)$ were time dependent. Since the behavior of the auxiliary system does not depend upon the state of the response system (and vice versa) the effects of time dependence of $\boldsymbol{\phi}(\cdot)$ would generally be uncorrelated, if the driving corresponds to an attractor of the autonomous system. Therefore, the stable regime of identical oscillations in the response and auxiliary systems would not be observable. At the same time if $\boldsymbol{\phi}(\cdot)$ did not satisfy property (2), it would be mapping points in the driving phase space onto continuous domains in the state spaces of the response and auxiliary systems. Once again, since the response and auxiliary systems are not coupled, the mappings inside the domains in these spaces would not generally be correlated and therefore the observation of the identical oscillations would not be possible. For the same reason the identity would be disrupted if there were no deterministic law for branch switching. Finally, if the transformation were not continuous (property (3)), then any small uncorrelated perturbations in the response and auxiliary systems would generally result in occasional finite scale deviations from the identity. In other words, the regime of identical oscillations would not be robust.

To summarize, the auxiliary system approach includes (i) construction of the auxiliary system, which is an exact replica of the response system and is driven by a signal from the driving system in the same fashion as the response system; (ii) demonstration of the local stability of the manifold of identical oscillations in the combined phase space of the response and auxiliary systems; and (iii) demonstration of robustness of the identity relationship with respect to small uncorrelated perturbations in response and auxiliary systems. When one can prove the local stability of this manifold and the robustness of the identical oscillations, the conclusion that follows is that in the combined phase space of the driving and response systems there exists an attractor that is the image of generalized synchronized chaotic oscillations.

In our laboratory experiments the local stability of the manifold of identical oscillations can be verified by means of observation of the regime of stable identical oscillations of the response and auxiliary systems. The presence of natural noise in any physical experiment ensures that the system does not stay in an unstable regime. Therefore, if one observes for long times the identical oscillations of the response and auxiliary systems, this indicates the stability of the synchronization manifold and the continuity of the transformation $\boldsymbol{\phi}(\cdot)$, which follows from the robustness of this regime.

III. AN EXAMPLE OF GENERALIZED SYNCHRONIZATION

Let us now come down from the general discussion of detecting synchronization by means of an auxiliary system to a specific example. We consider the generalized synchronization of chaotic oscillations in a three-dimensional Lorenz system when it is driven by a chaotic signal from a Rössler system. In this case the drive and the response systems are given by the following equations. For the drive system (Rössler)

$$\begin{aligned} \dot{x}_1(t) &= -[x_2(t)+x_3(t)], \\ \dot{x}_2(t) &= x_1(t)+0.2x_2(t), \\ \dot{x}_3(t) &= 0.2+x_3(t)[x_1(t)-\mu] \end{aligned} \tag{7}$$

and for the response system (Lorenz)

$$\begin{aligned} \dot{y}_1(t) &= \sigma[y_2(t)-y_1(t)]-g[y_1(t)-x_1(t)], \\ \dot{y}_2(t) &= -y_1(t)y_3(t)+ry_1(t)-y_2(t), \\ \dot{y}_3(t) &= y_1(t)y_2(t)-by_3(t). \end{aligned} \tag{8}$$

In the Rössler system we have $\mu=5.7$ and in the Lorenz system we have chosen $\sigma=16$, $b=4$, and $r=45.92$. The response system is coupled to the drive system only through the scalar forcing term $x_1(t)$. g characterizes the strength of the unidirectional coupling.

Obviously, in these coupled systems $\mathbf{x}(t)=\mathbf{y}(t)$ is just not possible. However, these systems can be synchronized in the generalized sense. To demonstrate this we introduce the auxiliary system (Lorenz)

$$\begin{aligned} \dot{z}_1(t) &= \sigma[z_2(t)-z_1(t)]-g[z_1(t)-x_1(t)], \\ \dot{z}_2(t) &= -z_1(t)z_3(t)+rz_1(t)-z_2(t), \\ \dot{z}_3(t) &= z_1(t)z_2(t)-bz_3(t), \end{aligned}$$

which is a replica of the response system (8), and show that the limit set of synchronized trajectories in the manifold $\mathbf{y}(t)=\mathbf{z}(t)$ can be stable to perturbations transverse to this manifold when the coupling is sufficiently strong.

We consider the linearized equations for perturbations transverse to the manifold $\mathbf{y}(t)=\mathbf{z}(t)$

$$\begin{aligned} \dot{\xi}_1(t) &= \sigma[\xi_2(t)-\xi_1(t)]-g\xi_1(t), \\ \dot{\xi}_2(t) &= -z_3(t)\xi_1(t)-z_1(t)\xi_3(t)+r\xi_1(t)-\xi_2(t), \\ \dot{\xi}_3(t) &= z_2(t)\xi_1(t)+z_1(t)\xi_2(t)-b\xi_3(t), \end{aligned}$$

where $\xi_a(t)=z_a(t)-y_a(t)$, $a=1,2,3$. The function

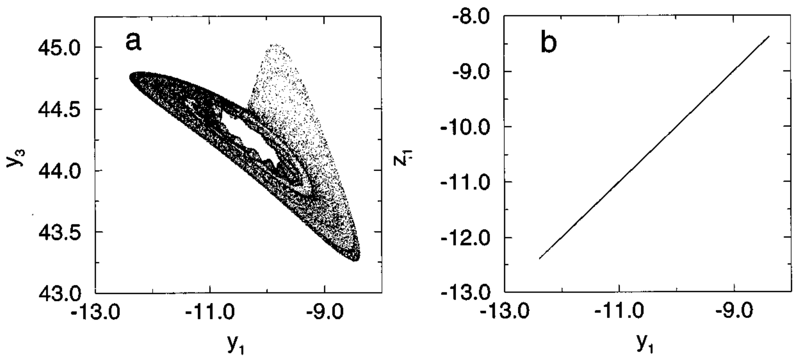

FIG. 1. Projections of synchronized attractors in the coupled Rössler and Lorenz systems. The coupling parameter is $g=10$ and the systems are coupled as indicated in the text. The Rössler system acts as the drive and the Lorenz system is the response. The projection of the response system onto the (y_1,y_3) plane is shown in (a). At this value of g the systems are synchronized. We can see this in the projection of the same attractor onto the (y_1,z_1) plane in (b).

$$V=\frac{1}{2}[4\xi_1^2(t)+\xi_2^2(t)+\xi_3^2(t)] \tag{9}$$

can be used as a Lyapunov function for the system if the value of the coupling parameter g satisfies

$$g>\left(\frac{1}{4}\sigma+r-z_3(t)\right)^2+\frac{z_2^2(t)}{b}-\sigma. \tag{10}$$

Since the values of $z_a(t)$ are bounded, this condition is satisfied when g is large enough. The boundedness of the attractors in the response and auxiliary systems under the bounded chaotic drive $x_1(t)$ can be easily shown for $g\to\infty$. In this case all trajectories of the auxiliary system are quickly attracted by the manifold of "slow" motions $z_1(t)\approx x_1(t)$ on which their further evolution is governed by the equations

$$\begin{aligned}\dot{z}_2(t)&=-x_1(t)z_3(t)+rx_1(t)-z_2(t),\\ \dot{z}_3(t)&=x_1(t)z_2(t)-bz_3(t).\end{aligned} \tag{11}$$

Now consider the positive function

$$V_p(t)=\frac{1}{2}[z_2^2(t)+z_3^2(t)]. \tag{12}$$

The time derivative of this is

$$\frac{dV_p(t)}{dt}=-bz_3^2(t)-[z_2(t)-rx_1(t)/2]^2+[rx_1(t)/2]^2. \tag{13}$$

It is easy to see from this that in regions of the (z_2,z_3) plane far from the origin, $\dot{V}_p(t)$ is negative. Therefore, far from the origin the distance to it, $\sqrt{z_2^2(t)+z_3^2(t)}$, decreases in time. This means that all trajectories in the (z_2,z_3) plane that begin far from the origin end up in a bounded domain centered at the origin. Therefore, oscillations of the response and auxiliary systems after transients die away are bounded and the condition given on the coupling g can be satisfied.

This shows that in the limit of strong coupling the attractor in the manifold $\mathbf{y}(t)=\mathbf{z}(t)$ is linearly stable to transverse perturbations and therefore the manifold of synchronized motions in the total phase space of the coupled Lorenz and Rössler systems is stable as well. Moreover, using the Lyapunov function (9) one can prove the robustness of the regime of identical oscillations in the response and auxiliary systems with respect to small uncorrelated perturbations added to the driving signals in each of these two systems. Thus, in the limit of strong coupling the coupled Rössler and Lorenz systems are synchronized in the generalized sense.

The generalized synchronization of chaos in these coupled systems also occurs at finite values of g. To demonstrate this we have two approaches. The first requires the computation of conditional Lyapunov exponents [6]. However, the negativity of the conditional Lyapunov exponents does not always guarantee the stability of synchronized motions in practical settings [12–15]. Instead we use an alternative approach that employs an auxiliary system.

We integrate all three systems (drive, response, and auxiliary), occasionally introducing small random fluctuations into the values of dynamical variables. The introduction of small perturbations into the system in numerical simulations conditions is required to ensure the stability of the synchronization manifold. The effects of such perturbations on the synchronized chaos and robustness issues are discussed in [16]. The results of the computer simulations at $g=10$ are presented in Fig. 1. Figs. 1(a) and 1(b) show the projections of the attractor from the nine-dimensional phase space onto the planes (y_1,y_3) and (y_1,z_1), respectively. One can see from the plot shown in Fig. 1(b) that the manifold $\mathbf{y}=\mathbf{z}$ is stable, and therefore the manifold of synchronized motions specified by $\mathbf{y}=\boldsymbol{\phi}(\mathbf{x})$ is stable as well, and the chaotic oscillations in the drive and response systems are synchronized. We also studied the (y_2,z_2) and (y_3,z_3) projections of the attractor. The plots for these projections look identical to the one in Fig. 1(b).

Although the numerical implementation of the auxiliary system test for systems described by ordinary differential equations (ODEs) may not seem to be more informative than computing the conditional Lyapunov exponents for the response system, there are two categories of problems where this method may be practically the only tool for detection of generalized synchronization. One category is the synchronization of chaos in time delay systems such as the Mackey-Glass system [17]. The other category includes synchronization of spatiotemporal chaos in spatially extended systems [1,2]. In each of these cases the concept of conditional Lyapunov exponents is not only not well developed, but also the computation of these exponents is a very cumbersome process. The auxiliary system method may be a big help in

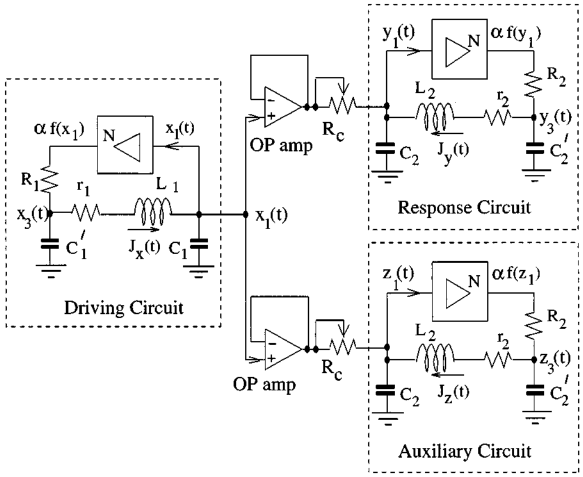

FIG. 2. Circuit diagram of the experiment with driving, response, and auxiliary nonlinear electrical circuits. The parameters values of the drive circuit are set to be $C'_1 = 230$ nF, $C_1 \approx 337$ nF, $L_1 \approx 140$ mH, $r_1 \approx 334\ \Omega$, and $R_1 \approx 4.21\ \text{k}\Omega$. The parameters values of the response and auxiliary circuits are $C'_2 \approx 225$ nF, $C_2 \approx 342$ nF, $L_2 \approx 145$ mH, $r_2 \approx 348\ \Omega$, and $R_2 \approx 4.97\ \text{k}\Omega$.

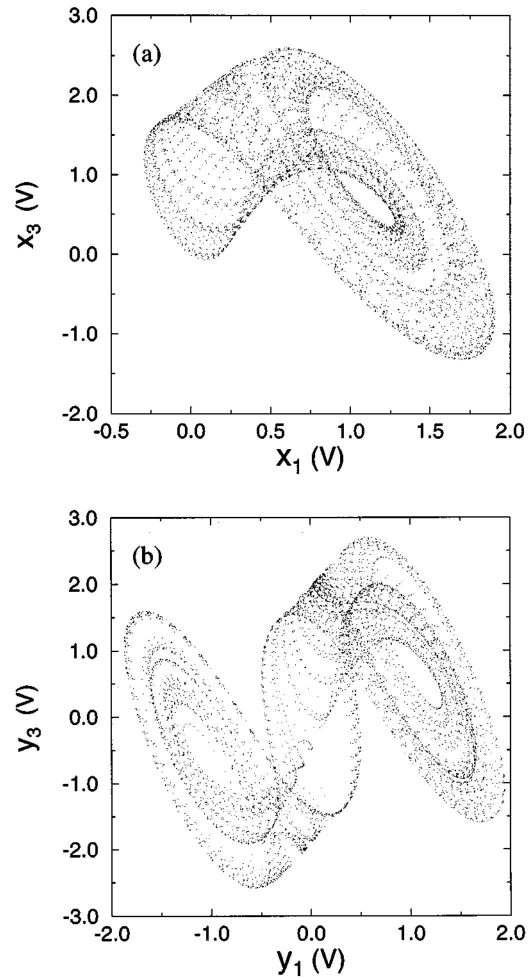

FIG. 3. Experimentally measured chaotic attractors of the (a) uncoupled drive and (b) response circuits. The parameter in the nonlinear converters N in the drive is $\alpha \approx 22.85$ and in the response it is $\alpha \approx 24.62$. (a) is the projection of the drive attractor onto the (x_1, x_3) plane. (b) is the projection of the response attractor onto the (y_1, y_3) plane. The attractors in the two systems are not the same, as one might expect as the systems are different.

general when one works with a system having a very large number of degrees of freedom since the estimation of Lyapunov exponents in such cases requires substantial computational power and the methods based on time series analysis [7, 9] may become unreliable.

However, even for the analysis of generalized synchronization in low-dimensional ODE systems this method has a certain appeal. Employing it is less complicated than computing conditional Lyapunov exponents and it allows one to detect *generalized* synchronization in a fashion reminiscent of the straightforward methods used for detection of identical synchronized chaotic oscillations. Also, as we already mentioned, in practice, the synchronization of chaos may break down even when all global conditional Lyapunov exponents are negative [12–15]. In this sense the auxiliary system test for the stability of synchronization manifold is more reliable.

IV. THE AUXILIARY SYSTEM APPROACH IN AN EXPERIMENT WITH ELECTRONIC CIRCUITS

To study the generalized synchronization of chaos in an experiment with electronic circuits we built two almost identical electronic circuits that were driven by a chaotic signal from a third circuit. The circuit diagram of the experiment is shown in Fig. 2. More details on the design of these chaotic circuits can be found elsewhere [16, 18]. We consider one of the driven circuits as the response circuit and the other as the auxiliary circuit. The chaotic signal generated by the drive circuit was applied to both the response and the auxiliary circuits through the resistors R_c. The strength of the coupling was controlled by the values of R_c in each circuit. This was adjusted to have the same value for both circuits.

In the experiment we tuned the parameters of the drive circuit to correspond to the regime of chaotic oscillations. The attractor is shown in Fig. 3(a) in a projection onto the (x_1, x_3) plane. The parameters of the response and the auxiliary circuits were tuned to values that, without coupling, namely, $R_c \to \infty$, lead these circuits to generate chaotic oscillations corresponding to the attractor shown in Fig. 3(b). In the figures the horizontal and vertical axes correspond to the voltages measured across the capacitors C_k and C'_k, respectively. ($k = 1$ and 2 for the drive and response, respectively.)

Synchronization of the chaotic oscillations was observed for values of the coupling with $R_c < 630\ \Omega$. It was easily detected with the analysis of the projections of the synchronization manifold onto the planes (y_1, z_1) and (y_3, z_3). For the synchronized oscillations the manifold is projected onto the diagonals $y_1 = z_1$ and $y_3 = z_3$ on these planes. These identities guarantee the identity of the currents $J_y(t) = J_z(t)$, which one can see in Fig. 2. Therefore the chaotic behaviors of the response and auxiliary circuits are identical. Since this regime of identical oscillations is observed in the presence of natural noise, this observation guarantees the stability of the synchronization manifold in the phase space of drive and response systems.

Synchronized behavior, observed with $R_c = 604\ \Omega$, is shown in Fig. 4. The synchronized chaotic attractor measured in the response circuit is presented in Fig. 4(a). The fact of the synchronization is confirmed by the stability of the ''diagonal manifold'' in the state space of the *response-auxiliary* system [see Fig. 4(b)], from which the stability of the manifold of synchronized motions in the phase space of the *drive-response* system follows. Thus the chaotic oscillations in drive and response circuits are synchronized. Looking at the projections of the synchronized chaotic attractors onto the plane of the variables (x_1, y_1) and (x_3, y_3), it becomes clear that oscillations in the driving and response circuits are not identical; see Fig. 4(c). Therefore, these circuits are synchronized only in the generalized sense.

Unsynchronized chaotic oscillations, measured with $R_c = 731\ \Omega$, are shown in Fig. 5. Although the projections of the measured attractors onto the variables of the drive and response circuits do not look much different from the previous case of synchronized behavior [compare Figs. 4(a) and 4(c) with Figs. 5(a) and 5(c)], the projection of these chaotic oscillations onto the plane (y_3, z_3) clearly indicate that these oscillations are not synchronized. Comparing different projections of the chaotic attractors in the drive-response and drive-auxiliary systems, we concluded that two pairs of systems evolve on the identical attractors. Therefore Fig. 5(b) can only indicate the loss of the synchronization.

V. CONCLUSION

Synchronization of nonlinear oscillators, whether they are evolving in a chaotic or regular fashion, can take many interesting forms. When the coupled systems are identical, the dynamical variables of the oscillators may tend toward identical motion and they may, under other circumstances, tend toward out of phase, but clearly synchronized, motions [3]. Synchronization may well occur when the oscillators are not identical, and several examples have been given here and in [7, 9]. In some sense the synchronization of nonlinear systems that are not identical is critical for the appearance of synchronized motions in real systems where precise identity of the systems is unlikely. The form that synchronization takes in realistic systems is likely to be richer than a precise identity between dynamical variables and we have addressed here and in earlier work [7] a class of generalized synchronized motions that enlarges one's view of this phenomenon.

In this paper we have worked with coupled systems that do not have mutual feedback but are organized as a drive system whose dynamical variables $\mathbf{x}(t)$ are ''inherited'' by a response system $\mathbf{y}(t)$ through some communications channel between them. The appearance of such drive-response systems is widespread in applications to communications.

Under general forms of coupling the drive to the response system, one expects that the dynamical variables of the response $\mathbf{y}(t)$, while functionally related to the drive, do not ''track'' the drive variables $\mathbf{x}(t)$ in any fashion. Predicting the response orbits from knowledge of the drive alone would not be possible in general.

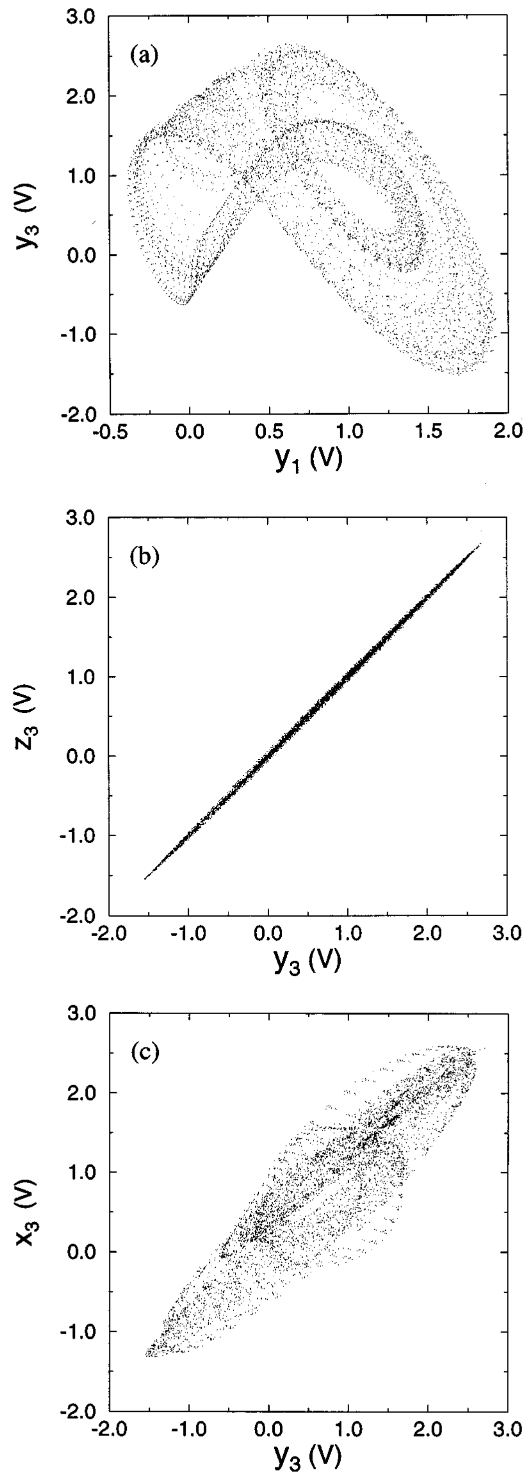

FIG. 4. Experimentally measured aspects of the synchronized chaotic attractor in the setup shown in Fig. 2. The coupling resistor was set at $R_c = 604\ \Omega$. (a) is the projection of the attractor onto the (y_1, y_3) plane, (b) is the projection of the attractor onto the (y_3, z_3) plane, and (c) is the projection of the attractor onto the (y_3, x_3) plane. From (b) we conclude that the systems are synchronized, while from (c) we can see that the oscillations in the drive and response systems are not identical and thus these systems are synchronized in the generalized sense.

When the systems become synchronized in the generalized sense, however, the situation alters in an essential fashion. The motion in the combined phase space of the drive and the response collapses in a stable way onto a mani-

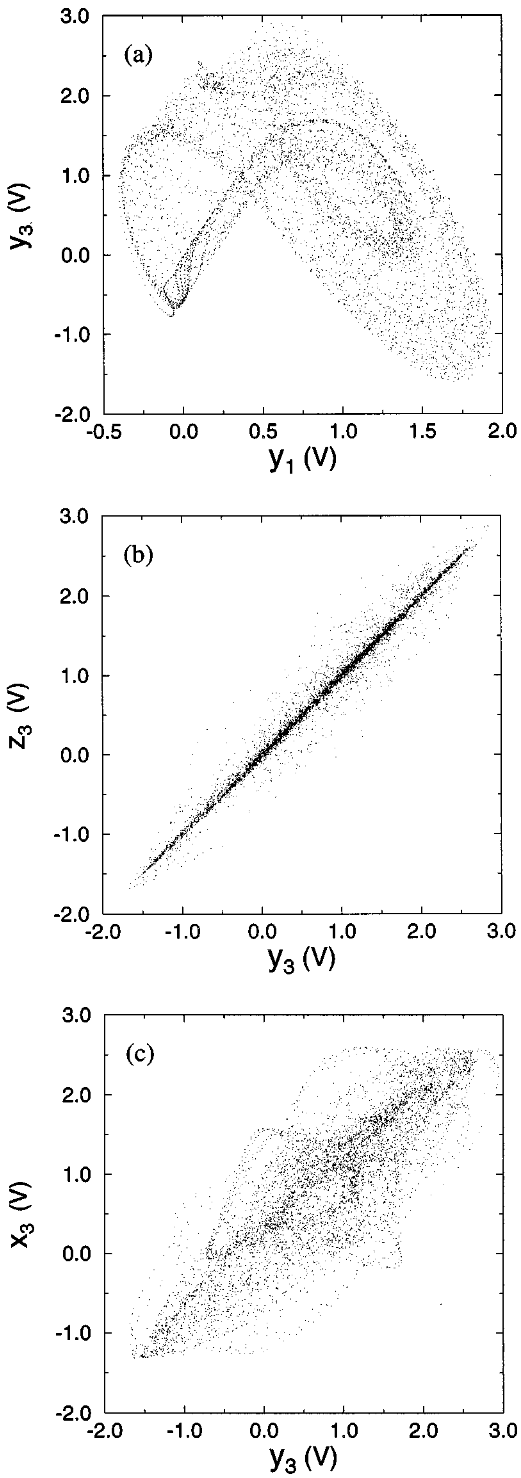

FIG. 5. Experimentally measured aspects of the unsynchronized chaotic attractor in the setup shown in Fig. 2. The coupling resistor was set at $R_c=731\ \Omega$. (a) is the projection of the attractor onto the (y_1,y_3) plane, (b) is the projection of the attractor onto the (y_3,z_3) plane, and (c) is the projection of the attractor onto the (y_3,x_3) plane. From (b) we conclude that the systems are not synchronized.

fold dictated by the synchronization relationship $\mathbf{y}(t)=\boldsymbol{\phi}(\mathbf{x}(t))$, and when orbits reach this synchronization manifold they remain there. When one has a finite basin of attraction within the joint phase space, orbits move on the stable synchronization manifold independently of the initial condition of the response system. Precisely how they traverse the manifold depends, of course, on the initial condition, but their presence there only requires the initial condition to lie in the basin of attraction. This is not different from familiar properties of orbits coming stably to some attractor.

The motion in either X or Y can be chaotic or regular. The connection between them $\mathbf{y}(t)=\boldsymbol{\phi}(\mathbf{x}(t))$ is what remains crucial and implies that one can predict from knowledge of the drive $\mathbf{x}(t)$ alone the dynamics of the response. Unfortunately, the relationship between the $\mathbf{x}(t)$ and the $\mathbf{y}(t)$ is generically complicated and unknown. This paper has noted that if we drive two identical response systems, one the original response system with variables $\mathbf{y}(t)$ and the other our auxiliary system with variables $\mathbf{z}(t)$, with the same input $\mathbf{x}(t)$ from the autonomous drive system, then we can identify the presence of the synchronization function $\boldsymbol{\phi}(\cdot)$ by observing the stable regime of identical oscillations in auxiliary and response systems $\mathbf{y}(t)=\mathbf{z}(t)$.

We demonstrated the usefulness of the auxiliary system approach both in a simulation using a Lorenz system driven by a Rössler system and more persuasively in an experiment using coupled non-linear circuits. In each case we were able to see synchronization of the generalized sort, $\mathbf{y}(t)=\boldsymbol{\phi}(\mathbf{x}(t))$, without the elaborate computation required [7, 9] without the auxiliary system.

The auxiliary system need not be implemented in actual circuitry, if circumstances do not allow. If measurements of the drive signal $\mathbf{x}(t)$ are well sampled, then using an accurate simulation of the response would allow one to compare the simulation, driven by $\mathbf{x}(t)$, to the measurements of a (possibly analog) response system. In this way, we anticipate that the observations we have made may be useful in real time recognition of synchronization between dynamical systems and thus lead to further investigation and potential exploitation of their synchrony.

As a final note we recall that if there are multiple basins of attraction for the coupled drive-response system, then the auxiliary system approach could fail. The response and auxiliary systems satisfy precisely the same differential equations and are driven by precisely the same drive dynamics $\mathbf{x}(t)$, so the only way their orbits could not track the same attractor would be when their initial conditions $\mathbf{y}(0)$ and $\mathbf{z}(0)$ lie in different basins of attraction. Although we did not encounter this in our numerical example or in our experiment, it is worth bearing in mind. A check on the problem, since such an occurrence may be difficult to distinguish from unsynchronized motion, would be to return to some of the numerical tools explored in [7, 9].

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[11] Clearly, in physical experiments and numerical simulations one can speak of continuity only up to a certain resolution. Therefore, from a practical point of view, any analysis can be limited to the consideration of only such discontinuities that are larger than the physical resolution scale.

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